UDC 62-50

ON THE INFORMATION AVAILABLE TO PLAYERS IN A DIFFERENTIAL GAME

PMM Vol. 36, №5, 1972, pp. 917-924

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(Received May 3, 1971)

We consider three possible statements of the problem of termination of a differential game from a given point. We derive sufficient conditions for the completion of a linear differential game under a significant discrimination of the pursuer.

1. Let the motion of a vector z in an n-dimensional Euclidean space R be described by the vector differential equation

$$dz/dt = f(z, u, v) \tag{1}$$

where $u \in P$ and $v \in Q$ are control parameters varying on sets P and Q which are compact in R. Regarding the right-hand side of Eq. (1) we assume that:

a) $f(z, u, v)$ is continuous in $(z, u, v) \in R \times P \times Q$;

b) the inequality

$$| f(z_1, u, v) - f(z_2, u, v) | \leqslant k | z_1 - z_2 |$$

where k is a constant depending only on C, is fulfilled for any $u \in P$, $v \in Q$ and for $z_1, z_2 \in R$, $|z_1| \leqslant C$, $|z_2| \leqslant C$;

c) there exists a constant B such that

$$| z \cdot f(z, u, v) | \leqslant B \quad (1 + | z |^2)$$

holds for all $z \in R$, $u \in P$, $v \in Q$;

d) the set $f(z, P, v)$ is convex for any $z \in R$, $v \in Q$. Furthermore, let a certain closed set M be specified in R. We say that the data listed above describe a differential game (1).

The measurable vector-valued functions $u^* = \{u(t), t \geqslant 0\}$, $v^* = \{v(t), t \geqslant 0\}$, satisfying the inclusions $u(t) \in P$, $v(t) \in Q$ for each t, are called the controls of the players U and V, respectively. The goal of the player U is to drive the point z onto set M, while player V seeks to prevent this. The game is completed when the vector z first hits onto M. We remark that when conditions (a) — (d) are fulfilled, for any $z_0 \in R$ $(0 \leqslant \tau \leqslant T)$ and for any pair of controls u^*, v^* defined on $[\tau, T]$, there exists, and is unique [1], a solution $z(t)$ $(\tau \leqslant t \leqslant T)$ of Eq. (1) with the initial condition $z(0) = z_0$ (i.e. a vector-valued function $z(t)$, absolutely continuous on $[\tau, T]$, satisfying Eq. (1) almost everywhere). The function $z(t)$ is called the motion and is denoted $z(t) = z(t; \tau, z_0, u^*, v^*, T)$. For fixed τ, T, v^* the set of motions is compact [2, 3]: if $z_i \to z_0$ as $i \to \infty$, then from any sequence of motions $z_i(t) = z(t; \tau, z_i, u_i^*, v^*, T)$ we can select a subsequence $z_{n_i}(t)$ converging, uniformly on $[\tau, T]$ to some motion $z(t; \tau, z_0, u^*, v^*, T)$. Uniform convergence on $[\tau, T]$ will be denoted by the symbol . . .

We say that game (1) can be completed from a point z_0 in time $t(z_0)$ if (whatever be the control v^* of player V) the player U can so construct his own control u^* that the point $z(t) = z(t; 0, z_0, u^*, v^*, t)$ hits onto the set M no later than in a time $t(z_0)$. As regards the information available to player U we assume here that each instant t he knows $z(t)$ and

(I) the ε-sprout of the control of player V, i.e. $v(s)$, $t \leqslant s \leqslant t+\varepsilon$;

(II) $v(s)$, $s \leqslant t$;

(III) he is forced to give his ε-sprout $u(s)$ $(t \leqslant s \leqslant t+\varepsilon)$, after which player V chooses the control $v(t)$.

In this paper we prove that statements (I) and (II) of the problem of terminating game (1) from a given point z_0 are, in a specific sense, equivalent. For this purpose we introduce an operator F_ε (an analog of the operator T_ε in [4]) and to the differential game (1) we apply the method of the authors of [4, 5] in combination with the constructions in [6]. The proofs of the assertions made are obtained by a formal replacement of T_ε by F_ε (the role of the lemma in Sect. 11 of [6] is here played by Lemma 1 proved below), and we omit them. Below we have pointed out the case when a certain time $T = T(z_0)$ of completion of game (1) from a given point z_0, determined for statement (I), is sufficient for its termination in the sense of statement (III).

2. Let ε be an arbitrary positive number. We define an operator F_ε: $2^R \rightarrow 2^R$ in the following manner: for any $X \subset R$ the point z_0 belongs to set $F_\varepsilon(X)$ if and only if (whatever be the control v^* of player V) we can find a control u^* of player U such that $z(\varepsilon) = z(\varepsilon; 0, z_0, u^*, v^*, \varepsilon) \in X$. We note the following properties of operator F_ε [4]:

1°. If $X_1 \subset X_2$, then $F_\varepsilon(X_1) \subset F_\varepsilon(X_2)$;

2°. $F_{\varepsilon_1}(F_{\varepsilon_2}(X)) \subset F_{\varepsilon_1+\varepsilon_2}(X)$;

3°. If X is closed, then $F_\varepsilon(X)$ is also closed;

4°. If $\{X_i\}_{i=1}^\infty$ is a sequence of closed sets such that $X_{i+1} \subset X_i$ $(i = 1, 2, \ldots)$, then

$$F_\varepsilon\left(\bigcap_{i=1}^\infty X_i\right) = \bigcap_{i=1}^\infty F_\varepsilon(X_i)$$

Let t be an arbitrary positive number. Every set $\omega_t = \{\tau_0, \tau_1, \ldots, \tau_m\}$ of real numbers $\tau_0 = 0 < \tau_1 < \ldots < \tau_m = t$ is called a partitioning of the interval $[0, t]$. We set $\delta_i = \tau_i - \tau_{i-1}$ $(i = 1, \ldots, m)$ and $|\omega_t| = \max \delta_i$. On the set of all partitionings of interval $[0, t]$ we introduce an order relation < by setting $\omega_t' < \omega_t''$ if and only if each point of partitioning ω_t' is a point of partitioning ω_t''. With every partitioning ω_t of interval $[0, t]$ we associate an operator F_{ω_t}: $2^R \rightarrow 2^R$ acting in the following manner:

$$F_{\omega_t}(X) = F_{\delta_m}(F_{\delta_{m-1}}(\ldots(F_{\delta_1}(X))\ldots)), \qquad X \subset R$$

From properties 1° – 4° it follows that:

5°. If X is closed, then $F_{\omega_t}(X)$ is also closed.

6°. If $\omega_t' < \omega_t''$, then $F_{\omega_t''}(X) \subset F_{\omega_t'}(X)$.

Lemma 1. Let X be closed, let $\omega_t = \{\tau_0, \tau_1, \ldots, \tau_m\}$ be an arbitrary partitioning of interval $[0, t]$, and let the sequence $\{\tau_1^k\}_{k=1}^\infty$ be such that $\tau_1 < \tau_1^k < \tau_2$; $\tau_1^k \rightarrow \tau_1$ as $k \rightarrow \infty$. Then

$$\bigcap_k F_{\omega_t^k}(X) \subset F_{\omega_t}(X), \qquad \omega_t^k = \{\tau_0, \tau_1^k, \tau_2, \ldots, \tau_m\}$$

For the proof of the lemma we need a number of definitions. For each $t > 0$, by U^t (V^t) we denote the set of all controls of player U (of player V), defined on $[0, t]$. Let $z_0 \in R$, $X \subset R$, $\omega_t = \{\tau_0, \tau_1, \ldots, \tau_m\}$ be an arbitrary partitioning of interval $[0, t]$, and D be a subset of V^t. The mapping $g = g(z_0, X, \omega_t, D): D \to U^t$ is called the ω_t-quasi-strategy at point z_0 relative to set X if:

a) whatever be $v_1^*, v_2^* \in D$ and $k \leqslant m$, from the equality $v_1(s) \equiv v_2(s)$, $0 \leqslant s \leqslant \tau_m - \tau_k$, there follows the equality $u_1(s) = g(v_1^*)(s) \equiv u_2(s) = g(v_2^*)(s)$ $(0 \leqslant s \leqslant \tau_m - \tau_k)$ ($u_i(s) = g(v_i^*)(s)$ is the value of the function $u_i^* = g(v_i^*) \in U^t$ at point s;

b) for any $v^* \in D$ there holds the inclusion $z(t) = z(t;\, 0, z_0, g(v^*), v^*, t) \in X$.

It is easy to verify the following:

Assertion 1. Let $z_0 \in R$, $X \subset R$. The inclusion $z_0 \in F_{\omega_t}(X)$ holds if and only if the quasi-strategy $g(z_0, X, \omega_t, V^t)$ exists.

We go on to prove Lemma 1. Let

$$z_0 \in \bigcap_k F_{\omega_t^k}(X)$$

Then by virtue of Assertion 1 there exists a sequence of quasi-strategies $g_k = g(z_0, X, \omega_t^k, V^t)$. The quasi-strategy $g = g(z_0, X, \omega_t, D)$ is called a c-quasi-strategy if for any $v^* \in D$ there exists a subsequence $\{g_{n_k}\}$ such that

$$z(s;\, 0, z_0, g_{n_k}(v^*), v^*, t) \Rightarrow z(s;\, 0, z_0, g(v^*), v^*, t), \quad s \in [0, t]$$

The set of c-quasi-strategies is nonempty. Indeed, let v_0^* be an arbitrary element of V^t. Then by virtue of the compactness of the set of motions, from the sequence of $z_k(s) = z(s;\, 0, z_0, g_k(v_0^*), v_0^*, t)$ we can single out a subsequence converging uniformly to some motion $z(s) = z(s;\, 0, z_0, u_0^*, v_0^*, t)$. The mapping $g = g(z_0, X, \omega_t, v_0^*)$, whose domain is the single point v_0^*, while $g(v_0^*) = u_0^*$, is obviously a c-quasi-strategy (the inclusion $z(t) \in X$ follows from the closedness of X).

On the set of c-quasi-strategies we introduce an order relation $<$ by setting $g_1(z_0, X, \omega_t, D_1) < g_2(z_0, X, \omega_t, D_2)$ if and only if $D_1 \subset D_2$ and for any $v^* \in D_1$ there holds $g_1(v^*)(s) \equiv g_2(v^*)(s)$, $0 \leqslant s \leqslant t$. It is easly verified that every linear ordering (see [7]) of the set F of c-quasi-strategies has a majorant. For example, a majorant is a c-quasi-strategy g^* with a domain $D^* = \bigcup D$ (the union of the right-hand side is taken over the whole domain of c-quasi-strategies occurring in F) such that for any $v^* \in D^*$ (and, consequently, $v \in D$ for some $g = g(z_0, X, \omega_t, D) \in F$) there is fulfilled $g^*(v^*) = g(v^*)$. In accordance with Zorn's lemma [7], in the set of c-quasi-strategies there exists a maximal element $g_0 = g_0(z, X, \omega_t, D_0)$. Let us show that $D_0 = V^t$, which, in accordance with Assertion 1, completes the proof of the lemma. We assume the contrary. Let $v_0^* \in V^t \setminus D_0$. We then define a mapping $g_* = g_*(z_0, X, \omega_t, D_0 \cup v_0^*)$ as follows: if $v^* \in D_0$, we set $g_*(v^*)(s) \equiv g_0(v^*)(s)$ $(0 \leqslant s \leqslant t)$. We define the function $g_*(v_0^*)$ as follows: by k_0 $(1 \leqslant k_0 \leqslant m)$ we denote the smallest positive integer for which the equality

$$v_0(s) \equiv v_{k_0}(s), \qquad 0 \leqslant s \leqslant \tau_m - \tau_{k_0} \tag{2}$$

is fulfilled for some $v_+^* = v_{k_0}^* \in D_0$. By the definition of a c-quasi-strategy there exists a subsequence $\{g^k = g_{n_k}\}$ such that

$$z(s;\, 0, z_0, g^k(v_+^*), v_+^*, t) \Rightarrow z(s) = z(s;\, 0, z_0, g_0(v_+^*), v_+^*, t) \quad s \in [0, t] \tag{3}$$

Case 1. $k_0 \geqslant 2$. Then from the sequence of

$$z_k^*(s) = z(s, 0, z_0, g^k(v_0^*), v_0^*, t)$$

by virtue of the compactness of the set of motions, we can choose a subsequence of $z_{k_j}^*(s)$ converging uniformly on $[0, t]$ to some motion $z^*(s) = z(s;\, 0, z_0, u_0^*, v_0^*, t)$. Since equality (2) is fulfilled on the interval $[0, \tau_m - \tau_{k_0}]$, then

$$g^k(v_0^*)(s) \equiv g^k(v_+^*)(s), \qquad 0 \leqslant s \leqslant \tau_m - \tau_{k_0}$$

and, consequently, by virtue of (3)

$$z^*_{k_j}(s) \Rightarrow z(s), \qquad 0 \leqslant s \leqslant \tau_m - \tau_{k_0}$$

whence $z^*(s) \equiv z(s)$ and $u_0(s) \equiv g_0(v_+^*)(s)$, $0 \leqslant s \leqslant \tau_m - \tau_{k_0}$. We complete the construction by setting $g_*(v_0^*)(s) \equiv u_0(s)$, $0 \leqslant s \leqslant t$.

Case 2. $k_0 = 1$. The functions v_+^* and v_0^* coincide on the interval $[0, \tau_m - \tau_1^{n_k}]$, therefore,

$$g^k(v_+^*)(s) \equiv g^k(v_0^*)(s), \qquad 0 \leqslant s \leqslant \tau_m - \tau_1^{n_k}$$

and, consequently, in accordance with (3)

$$z_k^*(s) = z(s; 0, z_0, g^k(v_0^*), v_0^*, t) \Rightarrow z(s; 0, z_0, g_0(v_+^*), v_+^*, t), \; s \in [0, \tau_m - \tau_1]$$

By choosing a subsequence $z_{k_j}^*(s)$ as needed, we can take it that

$$z^*_{k_j}(s) \Rightarrow z(s; 0, z_0, u_0^*, v_0^*, t), \qquad s \in [0, t]$$

where, by virtue of what we have said above, $u_0(s) \equiv g_0(v_+^*)(s)$ $0 \leqslant s \leqslant \tau_m - \tau_1$. We complete the construction by setting $g_*(v_0^*)(s) \equiv u_0(s)$ $0 \leqslant s \leqslant t$.

It is easily checked that in both cases the mapping g_* constructed is a c-quasi-strategy and $g_0 < g_*$, which contradicts the maximality of g_0. Thus, $D_0 = V^t$, which is what we required.

3. With each $t > 0$ we associate an operator F_t^*: $2^R \to 2^R$ in the following way:

$$F_t^*(X) = \bigcap_{\omega_t} F_{\omega_t}(X), \qquad X \subset R$$

(the intersection in the right-hand side is taken over all partitionings of interval $[0, t]$ We note the following properties of operator F_t^*:

7°. If X is closed, then $F_t^*(X)$ is also closed.

8°. Let X be closed and let $\{\omega_t^k\}_{k=1}^{\infty}$ be an arbitrary sequence of partitionings of interval $[0, t]$ such that $\omega_t^k < \omega_t^{k+1}$ $(k = 1, 2, \ldots)$ and $|\omega_t^k| \to 0$ as $k \to \infty$. Then

$$F_t^*(X) = \bigcap_{k=1}^{\infty} F_{\omega_t^k}(X)$$

9°. If X is closed and $0 < \varepsilon < t$, then $F_t^*(X) \subset F_\varepsilon(F_{t-\varepsilon}^*(X))$. From property 9° there directly ensues (see [4])

Theorem 1. Let $z_0 \in R$, $T \in (0, +\infty)$. Then if

$$z_0 \in F_T^*(M)$$

the differential game (1) can be completed from the point z_0 in time T in the sense of statement (I).

4. The mapping $g_t = g(z_0, M, t) : V^t \to U^t$, defined on all V^t, is called a t-strategy at point z_0 relative to M if:

a) whatever be v_1^*, $v_2^* \in V^t$, from the equality $v_1(s) \equiv v_2(s)$ $(0 \leqslant s \leqslant \varepsilon \leqslant t)$ follows the equality $g(v_1^*)(s) \equiv g(v_2^*)(s)$ $(0 \leqslant s \leqslant \varepsilon)$;

b) for any $v^* \in V^t$ the inclusion $z(t) = z(t; 0, z_0, g(v^*), v^*, t) \in M$ holds. We note that, obviously, every strategy $g = g(z_0, M, t)$ is an ω_t-quasi-strategy at point z_0 relative to M for any partitioning ω_t of interval $[0, t]$.

Theorem 2. Let $z_0 \in R$, $T \in (0, +\infty)$. Then if $z_0 \in F_T^*(M)$, there exists a T-strategy $g = g(z_0, M, T)$ such that the inclusion

$$z(t;\ 0,\ z_0,\ g(v^*),\ v^*,\ T) \in F^*_{T-t}(M), \quad 0 \leqslant t \leqslant T$$

holds for any control $v^* \in V^t$.

C o r o l l a r y. Under the hypotheses of Theorem 1, differential game (1) can be completed from the point z_0 in time T in the sense of statement (II).

Indeed, it is sufficient if at each instant t the player U sets his own control $u(t)$ equal to

$$u(t) = g(v_t^*)(t)$$

where g is the T-strategy given by Theorem 2,

$$v_t(s) \equiv v(s)\ \ 0 \leqslant s \leqslant t,\ \ v_t(s) \equiv v(t),\ t < s \leqslant T.$$

The inverse of Theorem 2 also proves to hold.

T h e o r e m 3. Let $z_0 \in R$, $T \in (0, +\infty)$, and let the T-strategy $g = g(z_0, M, T)$ exist. Then $z_0 \in F_T^*(M)$.

P r o o f. It is obviously sufficient to show that $z_0 \in F_{\omega_T}(M)$ for any partitioning ω_T of interval $[0, T]$. By virtue of Assertion 1 the latter is trivial because, as was noted above, every T-strategy is an ω_T-quasi-strategy.

5. For linear differential games, i.e. for games given by the equation [5]

$$dz / dt = Cz - u + v \tag{5}$$

the operator F_ε can be computed in explicit form. By direct calculation we verify that

$$F_\varepsilon(X) = \bigcap_{v^* \in V^\varepsilon} \left\{ e^{-\varepsilon C} \left[\left(X + \int_0^\varepsilon e^{rC} P\,dr \right) - \int_0^\varepsilon e^{rC} v(r)\,dr \right] \right\}$$

and, consequently,

$$F_t^*(X) = e^{-tC} W(t), \qquad W(t) = \int_{X,0}^{t} [e^{rC} P\,dr \,\underline{*}\, e^{rC} Q\,dr] \tag{6}$$

where $W(t)$ is the alternating integral from [5].

6. We proceed to study the possibility of the termination of a linear differential game in the sense of statement (III). We first recall certain concepts [5, 8]. Let $A \subset R$, $B \subset R$, and let α and β be real numbers. By definition, the set $\alpha A + \beta B$ consists of those, and only those, vectors $z \in R$ which are representable in the form $z = \alpha x + \beta y$ $(x \in A,\ y \in B)$. The set $D = A \,\underline{*}\, B$ of those, and only those, vectors $z \in R$ for which $z + B \subset A$, is called the geometric difference of sets A and B. It is easy to verify the following.

A s s e r t i o n 2. If A and B are convex and B is compact, then $(A + B) \,\underline{*}\, B = A$.

C o r o l l a r y. If A, B, C are convex, C is compact, and $A + C = B + C$, then $A = B$.

Let $A(t)$ be a compact convex set depending continuously (by inclusion) on $t \geqslant 0$. By the integral

$$\int_b^c A(\tau)\,d\tau, \qquad c \geqslant b \geqslant 0$$

we mean a compact [3] convex set consisting of those, and only those, $z \in R$ which can be represented in the form

$$z = \int_b^c a(\tau)\,d\tau$$

where $a^* = \{a(s),\ b \leqslant s \leqslant c\}$ is a measurable vector-valued function satisfying the inclusion $a(s) \in A(s)$ for every s. From the definition given it follows immediately that

$$\int_b^c A(\tau)\,d\tau + \int_c^d A(\tau)\,d\tau = \int_b^d A(\tau)\,d\tau \qquad (7)$$

Finally, we present without proof the following, easily verifiable –

Assertion 3. Let A be an ellipsoid of full dimension in R,

$$A = \left\{z : \sum_{i=1}^{n} \frac{(z^i)^2}{(a_i)^2} \leqslant 1\right\}$$

Then there exists a convex set $B \subset R$ such that

$$A + B = \frac{\alpha^2}{\beta} S_R$$

$$\alpha = \max_{1\leqslant i\leqslant n} a_i, \qquad \beta = \min_{1\leqslant i\leqslant n} a_i$$

where S_R is the unit sphere in R; moreover, if $A = A(t)$ depends continuously (by inclusion) on t, retaining full dimension in R, then $B = B(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$ also are continuous.

7. Let linear differential game be described by a vector differential equation (5) in which C is a constant square matrix of order n; let P and Q be convex compacta, and the terminal set M be representable in the form $M = M_0 + W_0$, where M_0 is a linear subspace of space R, W_0 is a convex compactum in the orthogonal complement L of M_0 in R. We denote the projection operator from R into L by π and the unit sphere in L by S. By L_P we denote the support plane to P (i.e. a set of the form $L_P = M_P + a$, where $a \in R$, M_P is a linear subspace of space R, such that the set $P - a$ belongs to M_P and has interior points therein). Let S_0 be the unit sphere in M_P.

We assume that the following conditions are fulfilled for game (5):

Condition 1. We can find $\lambda_0 > 0$ and a convex set $P' \subset R$ such that $P + P' = \lambda_0 S_0$.

Everywhere subsequently we agree to mean by r an arbitrary positive number. We consider the mapping $\Phi(r) = \pi e^{rC}: R \to L$ of space R into L.

Condition 2. The mapping $\Phi(r): M_P \to L$, treated as a mapping from M_P into L, is an "onto" mapping.

Lemma 2. Suppose the Conditions 1 and 2 have been satisfied for game (5). Then there exist a compact convex set $P(r) \subset L$, depending continuously (by inclusion) on r and a continuous positive function $\gamma(r)$ such that

$$\Phi(r)P + P(r) = \gamma(r)S, \quad r > 0 \qquad (8)$$

Proof. In accordance with Condition 1

$$\Phi(r)P + \Phi(r)P' = \lambda_0\Phi(r)S_0$$

From Condition 2 it follows that $\lambda_0\Phi(r)S_0$ is an ellipsoid of full dimension in L, depending continuously on r, and, consequently (Assertion 3),

$$\lambda_0\Phi(r)S_0 + B(r) = \gamma(r)S, \quad r > 0$$

where $B(r)$ and $\gamma(r)$ are continuous. We complete the proof of the lemma by setting $P(r) = \Phi(r)P' + B(r)$.

Let $t \geqslant 0$. We consider the set

$$W^*(t) = \left(W_0 + \int_0^t \Phi(r)\, Pdr\right) \overset{*}{-} \int_0^t \Phi(r)\, Qdr$$

We assume that the following conditions are fulfilled:

Condition 3. For any $t \geqslant 0$ the set $W^*(t)$ is nonempty and

$$W^*(t) + \int_0^t \Phi(r)\, Qdr = W_0 + \int_0^t \Phi(r)\, Pdr \tag{9}$$

Condition 4. For any $t > 0$ we can find $\lambda(t) > 0$ such that

$$W^*(t) = [W^*(t) \overset{*}{-} \lambda(t)\, S] + \lambda(t)\, S \tag{10}$$

It is easy to verify the following –

Assertion 4. Suppose that Condition 3 is satisfied for differential game (5). Then

$$W(t) = \int_{M,0}^t (e^{rC} Pdr \overset{*}{-} e^{rC} Qdr) = M_0 + W^*(t)$$

Thus, if the inclusion

$$\pi e^{TC} z_0 \in W^*(T) \tag{11}$$

is satisfied, then in accordance with Theorem 1 the linear differential game (5) can be completed from the point z_0 in time $T = T(z_0)$, where $T(z_0)$ is the minimum of all $T \geqslant 0$ for which inclusion (11) is fulfilled. This result is contained in the following theorem.

Theorem 4. Suppose that Conditions 1 – 4 are fulfilled for the linear differential game (5). Then, if inclusion (11) is fulfilled, game (5) can be completed from point z_0 in time $T = T(z_0)$ in the sense of statement (III).

Proof. For each $t > 0$ we denote by $\varepsilon(t)$ the largest positive number $\varepsilon \leqslant t/2$ (existing by virtue of Lemma 2) for which the inequality

$$\lambda(t) - \int_{t-\varepsilon}^t \gamma(r)\, dr \geqslant 0$$

is fulfilled. Let us show that for any $t > 0$ the following relation holds:

$$W^*(t) = \left[W^*(t - \varepsilon(t)) \overset{*}{-} \int_{t-\varepsilon(t)}^t \Phi(r)\, Qdr\right] + \int_{t-\varepsilon(t)}^t \Phi(r)\, Pdr \tag{12}$$

Indeed, in accordance with the corollary to Assertion 2, from equality (9) we have

$$W^*(t) + \int_{t-\varepsilon(t)}^t \Phi(r)\, Qdr = W^*(t - \varepsilon(t)) + \int_{t-\varepsilon(t)}^t \Phi(r)\, Pdr$$

whence, by virtue of (8),

$$W^*(t) + D + \int_{t-\varepsilon(t)}^t \Phi(r)\, Qdr = W^*(t - \varepsilon(t)) + \lambda(t)\, S$$

$$D = \int_{t-\varepsilon(t)}^t P(r)\, dr + \left(\lambda(t) - \int_{t-\varepsilon(t)}^t \gamma(r)\, dr\right) \cdot S$$

Therefore, on the basis of the corollary to Assertion 2 we obtain, using equality (10),

$$[W^*(t) \overset{*}{-} \lambda(t)\, S] + D = W^*(t - \varepsilon(t)) \overset{*}{-} \int_{t-\varepsilon(t)}^t \Phi(r)\, Qdr$$

Adding

$$\int_{t-\varepsilon(t)}^{t} \Phi(r)\, P dr$$

to both sides of this equality, we obtain the desired relation (12) (see the expression for D and formula (10)).

We set $T_0 = T_*(z_0)$, $\varepsilon_1 = \varepsilon(T(z_0))$. Since $\pi e^{T_0 C} z_* \in W^*(T_*)$, in accordance with (12) we can find a control $u_0^* = \{u_*(s),\ 0 \leqslant s \leqslant \varepsilon_1\}$ of player U such that

$$\pi e^{T_0 C} z_0 - \int_{T_0-\varepsilon_1}^{T_0} \pi e^{rC} u_0(T_0 - r)\, dr \in \left[W^*(T_0 - \varepsilon_1) \doteq \int_{T_0-\varepsilon_1}^{T_0} \pi e^{rC} Q dr\right]$$

Therefore, whatever be the control $v^* = \{v(s),\ 0 \leqslant s \leqslant \varepsilon_1\}$ of player V, for the point

$$z_1 = z(\varepsilon_1) = z(\varepsilon_1;\ 0, z_0, u_0^*, v^*, \varepsilon_1) = e^{\varepsilon_1 C}\left(z_0 - \int_0^{\varepsilon_1} e^{-sC}[u_0(s) - v(s)]\, ds\right)$$

we have

$$\pi e^{(T_0-\varepsilon_1)C} z_1 = \pi e^{T_0 C} z_0 - \int_{T_0-\varepsilon_1}^{T_0} \pi e^{rC} u_0(T_0 - r)\, dr + \int_{T_0-\varepsilon_1}^{T_0} \pi e^{rC} v(T_0 - r)\, dr \in W^*(T_0 - \varepsilon_1)$$

and, consequently, $T(z_1) \leqslant T_0 - \varepsilon_1$, whatever be the control of player V. Theorem 4 is proved if only we note that all the arguments presented above are applicable to the point $z_1 = z(\varepsilon_1)$, etc.

Pontriagin's verifying example [9] satisfies the hypotheses of Theorem 4.

The author thanks E. F. Mishchenko for guiding the work.

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Translated by N. H. C.